

Sufficient Conditions for Tuza's Conjecture on Packing and Covering Triangles*

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Abstract

Given a simple graph $G = (V, E)$, a subset of E is called a triangle cover if it intersects each triangle of G . Let $\nu_t(G)$ and $\tau_t(G)$ denote the maximum number of pairwise edge-disjoint triangles in G and the minimum cardinality of a triangle cover of G , respectively. Tuza conjectured in 1981 that $\tau_t(G)/\nu_t(G) \leq 2$ holds for every graph G . In this paper, using a hypergraph approach, we design polynomial-time combinatorial algorithms for finding small triangle covers. These algorithms imply new sufficient conditions for Tuza's conjecture on covering and packing triangles. More precisely, suppose that the set \mathcal{T}_G of triangles covers all edges in G . We show that a triangle cover of G with cardinality at most $2\nu_t(G)$ can be found in polynomial time if one of the following conditions is satisfied: (i) $\nu_t(G)/|\mathcal{T}_G| \geq \frac{1}{3}$, (ii) $\nu_t(G)/|E| \geq \frac{1}{4}$, (iii) $|E|/|\mathcal{T}_G| \geq 2$.

Keywords: Triangle cover, Triangle packing, Linear 3-uniform hypergraphs, Combinatorial algorithms

1 Introduction

Graphs considered in this paper are undirected, simple and finite (unless otherwise noted). Given a graph $G = (V, E)$ with vertex set $V(G) = V$ and edge set $E(G) = E$, for convenience, we often identify a triangle in G with its edge set. A subset of E is called a *triangle cover* if it intersects each triangle of G . Let $\tau_t(G)$ denote the minimum cardinality of a triangle cover of G , referred to as the *triangle covering number* of G . A set of pairwise edge-disjoint triangles in G is called a *triangle packing* of G . Let $\nu_t(G)$ denote the maximum cardinality of a triangle packing of G , referred to as the *triangle packing number* of G . It is clear that $1 \leq \tau_t(G)/\nu_t(G) \leq 3$ holds for every graph G . Our research is motivated by the following conjecture raised by Tuza [1] in 1981.

Conjecture 1.1 (Tuza's Conjecture [1]). $\tau_t(G)/\nu_t(G) \leq 2$ holds for every graph G .

To the best of our knowledge, the conjecture is still unsolved in general. If it is true, then the upper bound 2 is sharp as shown by K_4 and K_5 – the complete graphs of orders 4 and 5.

Related work. The only known universal upper bound smaller than 3 was given by Haxell [2], who shown that $\tau_t(G)/\nu_t(G) \leq 66/23 = 2.8695\dots$ holds for all graphs G . Haxell's proof [2] implies a polynomial-time algorithm for finding a triangle cover of cardinality at most $66/23$ times that of a maximal triangle packing. Other partial results on Tuza's conjecture concern with special classes of graphs.

Tuza [3] proved his conjecture holds for planar graphs, K_5 -free chordal graphs and graphs with n vertices and at least $7n^2/16$ edges. The proof for planar graphs [3] gives an elegant polynomial-time algorithm for

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finding a triangle cover in planar graphs with cardinality at most twice that of a maximal triangle packing. The validity of Tuza's conjecture on the class of planar graphs was later generalized by Krivelevich [4] to the class of graphs without $K_{3,3}$ -subdivision. Haxell and Kohayakawa [5] showed that $\tau_t(G)/\nu_t(G) \leq 2 - \epsilon$ for tripartite graphs G , where $\epsilon > 0.044$. Haxell, Kostochka and Thomasse [6] proved that every K_4 -free planar graph G satisfies $\tau_t(G)/\nu_t(G) \leq 1.5$.

Regarding the tightness of the conjectured upper bound 2, Tuza [3] noticed that infinitely many graphs G attain the conjectured upper bound $\tau_t(G)/\nu_t(G) = 2$. Cui, Haxell and Ma [7] characterized planar graphs G satisfying $\tau_t(G)/\nu_t(G) = 2$; these graphs are edge-disjoint unions of K_4 's plus possibly some vertices and edges that are not in triangles. Baron and Kahn [8] proved that Tuza's conjecture is asymptotically tight for dense graphs.

Fractional and weighted variants of Conjecture 1.1 were studied in literature. Krivelevich [4] proved two fractional versions of the conjecture: $\tau_t(G) \leq 2\nu_t^*(G)$ and $\tau_t^*(G) \leq 2\nu_t(G)$, where $\tau_t^*(G)$ and $\nu_t^*(G)$ are the values of an optimal fractional triangle cover and an optimal fractional triangle packing of G , respectively. The result was generalized by Chapuy et al. [9] to the weighted version, which amounts to packing and covering triangles in multigraphs G_w (obtained from G by adding multiple edges). The authors [9] showed that $\tau(G_w) \leq 2\nu^*(G_w) - \sqrt{\nu^*(G_w)/6} + 1$ and $\tau^*(G_w) \leq 2\nu(G_w)$; the arguments imply an LP-based 2-approximation algorithm for finding a minimum weighted triangle cover.

Our contributions. Along a different line, we establish new sufficient conditions for validity of Tuza's conjecture by comparing the triangle packing number, the number of triangles and the number of edges. Given a graph G , we use $\mathcal{T}_G = \{E(T) : T \text{ is a triangle in } G\}$ to denote the set consisting of the (edge sets of) triangles in G . Without loss of generality, we focus on the graphs in which every edge is contained in some triangle. These graphs are called *irreducible*.

Theorem 1.2. *Let $G = (V, E)$ be an irreducible graph. Then a triangle cover of G with cardinality at most $2\nu_t(G)$ can be found in polynomial time, which implies $\tau_t(G) \leq 2\nu_t(G)$, if one of the following conditions is satisfied: (i) $\nu_t(G)/|\mathcal{T}_G| \geq \frac{1}{3}$, (ii) $\nu_t(G)/|E| \geq \frac{1}{4}$, (iii) $|E|/|\mathcal{T}_G| \geq 2$.*

The primary idea behind the theorem is simple: any one of conditions (i) – (iii) allows us to remove at most $\nu_t(G)$ edges from G to make the resulting graph G' satisfy $\tau_t(G') = \nu_t(G')$; the removed edges and the edges in a minimum triangle cover of G' form a triangle cover of G with size at most $\nu_t(G) + \nu_t(G') \leq 2\nu_t(G)$. The idea is realized by establishing new results on linear 3-uniform hypergraphs (see Section 2); the most important one states that such a hypergraphs could be made acyclic by removing a number of vertices that is no more than a third of the number of its edges. A key observation here is that hypergraph (E, \mathcal{T}_G) is linear and 3-uniform.

To show the qualities of conditions (i) – (iii) in Theorem 1.2, we obtain the following result which complements to the constants $\frac{1}{3}$, $\frac{1}{4}$ and 2 in these conditions with $\frac{1}{4}$, $\frac{1}{5}$ and $\frac{3}{2}$, respectively.

Theorem 1.3. *Tuza's conjecture holds for every graph if there exists some real $\delta > 0$ such that Tuza's conjecture holds for every irreducible graph G satisfying one of the following properties: (i') $\nu_t(G)/|\mathcal{T}_G| \geq \frac{1}{4} - \delta$, (ii') $\nu_t(G)/|E| \geq \frac{1}{5} - \delta$, (iii') $|E|/|\mathcal{T}_G| \geq \frac{3}{2} - \delta$.*

We also investigate Tuza's conjecture on classical Erdős-Rényi random graph $\mathcal{G}(n, p)$, and prove that $\Pr[\tau_t(G)/\nu_t(G) \leq 2] = 1 - o(1)$ provided $G \in \mathcal{G}(n, p)$ and $p > \sqrt{3}/2$.

It is worthwhile pointing out that strengthening Theorem 1.2, our arguments actually establish stronger results for linear 3-uniform hypergraphs (see Theorem 4.1).

The rest of paper is organized as follows. Section 2 proves theoretical and algorithmic results on linear 3-uniform hypergraphs concerning feedback sets, which are main technical tools for establishing new sufficient conditions for Tuza's conjecture in Section 3. Section 4 concludes the paper with extensions and future research directions.

2 Hypergraphs

This section develops hypergraph tools for studying Tuza's conjecture. The theoretical and algorithmic results are of interest in their own right.

Let $\mathcal{H} = (\mathcal{V}, \mathcal{E})$ be a hypergraph with vertex set \mathcal{V} and edge set \mathcal{E} . For convenience, we use $\|\mathcal{H}\|$ to denote the number $|\mathcal{E}|$ of edges in \mathcal{H} . If hypergraph $\mathcal{H}' = (\mathcal{V}', \mathcal{E}')$ satisfies $\mathcal{V}' \subseteq \mathcal{V}$ and $\mathcal{E}' \subseteq \mathcal{E}$, we call \mathcal{H}' a *sub-hypergraph* of \mathcal{H} , and write $\mathcal{H}' \subseteq \mathcal{H}$. For each $v \in \mathcal{V}$, the *degree* $d_{\mathcal{H}}(v)$ is the number of edges in \mathcal{E} that contain v . We say v is an *isolated vertex* of \mathcal{H} if $d_{\mathcal{H}}(v) = 0$. Let $k \in \mathbb{N}$ be a positive integer, hypergraph \mathcal{H} is called *k-regular* if $d_{\mathcal{H}}(u) = k$ for each $u \in \mathcal{V}$, and *k-uniform* if $|e| = k$ for each $e \in \mathcal{E}$. Hypergraph \mathcal{H} is *linear* if $|e \cap f| \leq 1$ for any pair of distinct edges $e, f \in \mathcal{E}$.

A vertex-edge alternating sequence $v_1 e_1 v_2 \dots v_k e_k v_{k+1}$ of \mathcal{H} is called a *path* (of *length* k) between v_1 and v_{k+1} if $v_1, v_2, \dots, v_{k+1} \in \mathcal{V}$ are distinct, $e_1, e_2, \dots, e_k \in \mathcal{E}$ are distinct, and $\{v_i, v_{i+1}\} \subseteq e_i$ for each $i \in [k] = \{1, \dots, k\}$. We consider each vertex of \mathcal{H} as a path of length 0. Hypergraph \mathcal{H} is said to be *connected* if there is a path between any pair of distinct vertices in \mathcal{H} . A maximal connected sub-hypergraph of \mathcal{H} is called a *component* of \mathcal{H} . Obviously, \mathcal{H} is connected if and only if it has only one component.

A vertex-edge alternating sequence $\mathcal{C} = v_1 e_1 v_2 e_2 \dots v_k e_k v_1$, where $k \geq 2$, is called a *cycle* (of length k) if $v_1, v_2, \dots, v_k \in \mathcal{V}$ are distinct, $e_1, e_2, \dots, e_k \in \mathcal{E}$ are distinct, and $\{v_i, v_{i+1}\} \subseteq e_i$ for each $i \in [k]$, where $v_{k+1} = v_1$. We consider the cycle \mathcal{C} as a sub-hypergraph of \mathcal{H} with vertex set $\cup_{i \in [k]} e_i$ and edge set $\{e_i : i \in [k]\}$. For any $\mathcal{S} \subset \mathcal{V}$ (resp. $\mathcal{S} \subset \mathcal{E}$), we write $\mathcal{H} \setminus \mathcal{S}$ for the sub-hypergraph of \mathcal{H} obtained from \mathcal{H} by deleting all vertices in \mathcal{S} and all edges incident with some vertices in \mathcal{S} (resp. deleting all edges in \mathcal{E} and keeping vertices). If \mathcal{S} is a singleton set $\{s\}$, we write $\mathcal{H} \setminus s$ instead of $\mathcal{H} \setminus \{s\}$. For any $\mathcal{S} \subseteq 2^{\mathcal{V}}$, the hypergraph $(\mathcal{V}, \mathcal{E} \cup \mathcal{S})$ is often written as $\mathcal{H} \uplus \mathcal{S}$, and as $\mathcal{H} \oplus \mathcal{S}$ if $\mathcal{S} \cap \mathcal{E} = \emptyset$.

A vertex (resp. edge) subset of \mathcal{H} is called a *feedback vertex set* or FVS (resp. *feedback edge set* or FES) of \mathcal{H} if it intersects the vertex (resp. edge) set of every cycle of \mathcal{H} . A vertex subset of \mathcal{H} is called a *transversal* of \mathcal{H} if it intersects every edge of \mathcal{H} . Let $\tau_c^v(\mathcal{H})$, $\tau_c^e(\mathcal{H})$ and $\tau(\mathcal{H})$ denote, respectively, the minimum cardinalities of a FVS, a FES, and a transversal of \mathcal{H} . A *matching* of \mathcal{H} is a nonempty set of pairwise disjoint edges of \mathcal{H} . Let $\nu(\mathcal{H})$ denote the maximum cardinality of a matching of \mathcal{H} . It is easy to see that $\tau_c^v(\mathcal{H}) \leq \tau_c^e(\mathcal{H})$, $\tau_c^v(\mathcal{H}) \leq \tau(\mathcal{H})$ and $\nu(\mathcal{H}) \leq \tau(\mathcal{H})$. Our discussion will frequently use the trivial observation that if no cycle of \mathcal{H} contains any element of some subset \mathcal{S} of $\mathcal{V} \cup \mathcal{E}$, then \mathcal{H} and $\mathcal{H} \setminus \mathcal{S}$ have the same set of FVS's, and $\tau_c^v(\mathcal{H}) = \tau_c^v(\mathcal{H} \setminus \mathcal{S})$. The following theorem is one of main contributions of this paper.

Theorem 2.1. *Let \mathcal{H} be a linear 3-uniform hypergraph. Then $\tau_c^v(\mathcal{H}) \leq \|\mathcal{H}\|/3$.*

Proof. Suppose that the theorem failed. We take a counterexample $\mathcal{H} = (\mathcal{V}, \mathcal{E})$ with $\tau_c^v(\mathcal{H}) > |\mathcal{E}|/3$ such that $\|\mathcal{H}\| = |\mathcal{E}|$ is as small as possible. Obviously $|\mathcal{E}| \geq 3$. Without loss of generality, we can assume that \mathcal{H} has no isolated vertices. Since \mathcal{H} is linear, any cycle in \mathcal{H} is of length at least 3.

If there exists $e \in \mathcal{E}$ which does not belong to any cycle of \mathcal{H} , then $\tau_c^v(\mathcal{H}) = \tau_c^v(\mathcal{H} \setminus e)$. The minimality of $\mathcal{H} = (\mathcal{V}, \mathcal{E})$ implies $\tau_c^v(\mathcal{H} \setminus e) \leq (|\mathcal{E}| - 1)/3$, giving $\tau_c^v(\mathcal{H}) < |\mathcal{E}|/3$, a contradiction. So we have

- (1) Every edge in \mathcal{E} is contained in some cycle of \mathcal{H} .

If there exists $v \in \mathcal{V}$ with $d_{\mathcal{H}}(v) \geq 3$, then $\tau_c^v(\mathcal{H} \setminus v) \leq (|\mathcal{E}| - d_{\mathcal{H}}(v))/3 \leq (|\mathcal{E}| - 3)/3$, where the first inequality is due to the minimality of \mathcal{H} . Given a minimum FVS \mathcal{S} of $\mathcal{H} \setminus v$, it is clear that $\mathcal{S} \cup \{v\}$ is a FVS of \mathcal{H} with size $|\mathcal{S}| + 1 = \tau_c^v(\mathcal{H} \setminus v) + 1 \leq |\mathcal{E}|/3$, a contradiction to $\tau_c^v(\mathcal{H}) > |\mathcal{E}|/3$. So we have

- (2) $d_{\mathcal{H}}(v) \leq 2$ for all $v \in \mathcal{V}$.

Suppose that there exists $v \in \mathcal{V}$ with $d_{\mathcal{H}}(v) = 1$. Let $e_1 \in \mathcal{E}$ be the unique edge that contains v . Recall from (1) that e_1 is contained in a cycle $\mathcal{C} = v_1 e_1 v_2 e_2 v_3 \dots e_k v_1$, where $k \geq 3$. By (2), we have $d_{\mathcal{H}}(v_i) = 2$ for all $i \in [k]$. In particular $d_{\mathcal{H}}(v_1) = d_{\mathcal{H}}(v_2) = 2 > d_{\mathcal{H}}(v)$ implies $v \notin \{v_1, v_2\}$, and in turn $v_1, v_2, v \in e_1$ enforces $e_1 = \{v_1, v, v_2\}$. Let \mathcal{S} be a minimum FVS of $\mathcal{H}' = \mathcal{H} \setminus \{e_1, e_2, e_3\}$. It follows from (2) that

$$\mathcal{H} \setminus v_3 \subseteq \mathcal{H} \setminus \{e_2, e_3\} = \mathcal{H}' \oplus e_1,$$

and in $\mathcal{H}' \oplus e_1$, edge e_1 intersects at most one other edge, and therefore is not contained by any cycle. Thus \mathcal{S} is a FVS of $\mathcal{H}' \oplus e_1$, and hence a FVS of $\mathcal{H} \setminus v_3$, implying that $\{v_3\} \cup \mathcal{S}$ is a FVS of \mathcal{H} . We deduce that $|\mathcal{E}|/3 < \tau_c^\nu(\mathcal{H}) \leq |\{v_3\} \cup \mathcal{S}| \leq 1 + |\mathcal{S}|$. Therefore $\tau_c^\nu(\mathcal{H}') = |\mathcal{S}| > (|\mathcal{E}| - 3)/3 = \|\mathcal{H}'\|/3$ shows a contradiction to the minimality of \mathcal{H} . Hence the vertices of \mathcal{H} all have degree at least 2, which together with (2) gives

(3) \mathcal{H} is 2-regular.

Let $\mathcal{C} = (\mathcal{V}_c, \mathcal{E}_c) = v_1 e_1 v_2 e_2 \dots v_k e_k v_1$ be a shortest cycle in \mathcal{H} , where $k \geq 3$. For each $i \in [k]$, suppose that $e_i = \{v_i, u_i, v_{i+1}\}$, where $v_{k+1} = v_1$.

Because \mathcal{C} is a shortest cycle, for each pair of distinct indices $i, j \in [k]$, we have $e_i \cap e_j = \emptyset$ if and only if e_i and e_j are not adjacent in \mathcal{C} , i.e., $|i - j| \notin \{1, k - 1\}$. This fact along with the linearity of \mathcal{H} says that $v_1, v_2, \dots, v_k, u_1, u_2, \dots, u_k$ are distinct. By (3), each u_i is contained in a unique edge $f_i \in \mathcal{E} \setminus \mathcal{E}_c$, $i \in [k]$. We distinguish among three cases depending on the values of $k \pmod{3}$. In each case, we construct a proper sub-hypergraph \mathcal{H}' of \mathcal{H} with $\|\mathcal{H}'\| < \|\mathcal{H}\|$ and $\tau_c^\nu(\mathcal{H}') > \|\mathcal{H}'\|/3$ which shows a contradiction to the minimality of \mathcal{H} .

CASE 1. $k \equiv 0 \pmod{3}$: Let \mathcal{S} be a minimum FVS of $\mathcal{H}' = \mathcal{H} \setminus \mathcal{E}_c$. Setting $\mathcal{V}_* = \{v_i : i \equiv 0 \pmod{3}, i \in [k]\}$ and $\mathcal{E}_* = \{e_i : i \equiv 1 \pmod{3}, i \in [k]\}$, it follows from (3) that

$$\mathcal{H} \setminus \mathcal{V}_* \subseteq (\mathcal{H} \setminus \mathcal{E}_c) \oplus \mathcal{E}_* = \mathcal{H}' \oplus \mathcal{E}_*,$$

and in $\mathcal{H}' \oplus \mathcal{E}_*$, each edge in \mathcal{E}_* intersects exactly one other edge, and therefore is not contained by any cycle. Thus $(\mathcal{H}' \oplus \mathcal{E}_*) \setminus \mathcal{S}$ is also acyclic, so is $(\mathcal{H} \setminus \mathcal{V}_*) \setminus \mathcal{S}$, saying that $\mathcal{V}_* \cup \mathcal{S}$ is a FVS of \mathcal{H} . We deduce that $|\mathcal{E}|/3 < \tau_c^\nu(\mathcal{H}) \leq |\mathcal{V}_* \cup \mathcal{S}| \leq k/3 + |\mathcal{S}|$. Therefore $\tau_c^\nu(\mathcal{H}') = |\mathcal{S}| > (|\mathcal{E}| - k)/3 = \|\mathcal{H}'\|/3$ shows a contradiction.

CASE 2. $k \equiv 1 \pmod{3}$: Consider the case where $f_1 \neq f_3$ or $f_2 \neq f_4$. Relabeling the vertices and edges if necessary, we may assume without loss of generality that $f_1 \neq f_3$. Let \mathcal{S} be a minimum FVS of $\mathcal{H}' = \mathcal{H} \setminus (\mathcal{E}_c \cup \{f_1, f_3\})$. Set $\mathcal{V}_* = \emptyset$, $\mathcal{E}_* = \emptyset$ if $k = 4$ and $\mathcal{V}_* = \{v_i : i \equiv 0 \pmod{3}, i \in [k] - [3]\}$, $\mathcal{E}_* = \{e_i : i \equiv 1 \pmod{3}, i \in [k] - [6]\}$ otherwise. In any case we have $|\mathcal{V}_*| = (k - 4)/3$ and

$$\mathcal{H} \setminus (\{u_1, u_3\} \cup \mathcal{V}_*) \subseteq (\mathcal{H} \setminus (\mathcal{E}_c \cup \{f_1, f_3\})) \oplus (\{e_2, e_4\} \cup \mathcal{E}_*) = \mathcal{H}' \oplus (\{e_2, e_4\} \cup \mathcal{E}_*).$$

Note from (3) that in $\mathcal{H}' \oplus (\{e_2, e_4\} \cup \mathcal{E}_*)$, each edge in $\{e_2, e_4\} \cup \mathcal{E}_*$ can intersect at most one other edge, and therefore is not contained by any cycle. Thus $(\mathcal{H}' \oplus (\{e_2, e_4\} \cup \mathcal{E}_*)) \setminus \mathcal{S}$ is also acyclic, so is $(\mathcal{H} \setminus (\{u_1, u_3\} \cup \mathcal{V}_*)) \setminus \mathcal{S}$. Thus $\{u_1, u_3\} \cup \mathcal{V}_* \cup \mathcal{S}$ is a FVS of \mathcal{H} , and $|\mathcal{E}|/3 < \tau_c^\nu(\mathcal{H}) \leq |\{u_1, u_3\} \cup \mathcal{V}_* \cup \mathcal{S}| \leq 2 + |\mathcal{V}_*| + |\mathcal{S}| = (k + 2)/3 + |\mathcal{S}|$. This gives $\tau_c^\nu(\mathcal{H}') = |\mathcal{S}| > (|\mathcal{E}| - k - 2)/3 = \|\mathcal{H}'\|/3$, a contradiction.

Consider the case where $f_1 = f_3$ and $f_2 = f_4$. As u_1, u_2, u_3, u_4 are distinct and $|f_1| = |f_2| = 3$, we have $f_1 \neq f_2$. Observe that $u_1 e_1 v_2 e_2 v_3 e_3 u_3 f_3 u_1$ is a cycle in \mathcal{H} of length 4. The minimality of k enforces $k = 4$. Therefore $\mathcal{E}_c \cup \{f_1, f_2\}$ consist of 6 distinct edges. Let \mathcal{S} be a minimum FVS of $\mathcal{H}' = \mathcal{H} \setminus (\mathcal{E}_c \cup \{f_1, f_2\})$. It follows from (3) that

$$\mathcal{H} \setminus \{u_2, u_4\} \subseteq (\mathcal{H} \setminus (\mathcal{E}_c \cup \{f_1, f_2\})) \oplus \{e_1, e_3, f_1\} = \mathcal{H}' \oplus \{e_1, e_3, f_1\}.$$

In $\mathcal{H}' \oplus \{e_1, e_3, f_1\}$, both e_1 and e_3 intersect only one other edge, which is f_1 , and any cycle through f_1 must contain e_1 or e_3 . It follows that none of e_1, e_3, f_1 is contained by a cycle of $\mathcal{H}' \oplus \{e_1, e_3, f_1\}$. Thus $(\mathcal{H}' \oplus \{e_1, e_3, f_1\}) \setminus \mathcal{S}$ is acyclic, so is $(\mathcal{H} \setminus \{u_2, u_4\}) \setminus \mathcal{S}$, saying that $\{u_2, u_4\} \cup \mathcal{S}$ is a FVS of \mathcal{H} . Hence $|\mathcal{E}|/3 < \tau_c^\nu(\mathcal{H}) \leq |\{u_2, u_4\} \cup \mathcal{S}| \leq 2 + |\mathcal{S}|$. In turn $\tau_c^\nu(\mathcal{H}') = |\mathcal{S}| > (|\mathcal{E}| - 6)/3 = \|\mathcal{H}'\|/3$ shows a contradiction.

CASE 3. $k \equiv 2 \pmod{3}$: Let \mathcal{S} be a minimum FVS of $\mathcal{H}' = \mathcal{H} \setminus (\mathcal{E}_c \cup \{f_1\})$. Setting $\mathcal{V}_* = \{v_i : i \equiv 1 \pmod{3}, i \in [k] - [3]\}$ and $\mathcal{E}_* = \{e_i : i \equiv 2 \pmod{3}, i \in [k]\}$, we have $|\mathcal{V}_*| = (k - 2)/3$ and

$$\mathcal{H} \setminus (\{u_1\} \cup \mathcal{V}_*) \subseteq (\mathcal{H} \setminus (\mathcal{E}_c \cup \{f_1\})) \oplus \mathcal{E}_* = \mathcal{H}' \oplus \mathcal{E}_*$$

In $\mathcal{H}' \oplus \mathcal{E}_*$, each edge in \mathcal{E}_* intersects at most one other edge, and therefore is not contained by any cycle. Thus $(\mathcal{H}' \oplus \mathcal{E}_*) \setminus \mathcal{S}$ is acyclic, so is $(\mathcal{H} \setminus (\{u_1\} \cup \mathcal{V}_*)) \setminus \mathcal{S}$. Hence $\{u_1\} \cup \mathcal{V}_* \cup \mathcal{S}$ is a FVS of \mathcal{H} , yielding $|\mathcal{E}|/3 < \tau_c^\vee(\mathcal{H}) \leq |\{u_1\} \cup \mathcal{V}_* \cup \mathcal{S}| \leq 1 + (k-2)/3 + |\mathcal{S}|$ and a contradiction $\tau_c^\vee(\mathcal{H}') = |\mathcal{S}| > (|\mathcal{E}| - k - 1)/3 = \|\mathcal{H}'\|/3$.

The combination of the above three cases complete the proof. \square

We remark that the upper bound $\|\mathcal{H}\|/3$ in Theorem 2.1 is best possible. See Figure 1 for illustrations of five 3-uniform linear hypergraphs attaining the upper bound. It is easy to prove that the maximum degree of every extremal hypergraph (those \mathcal{H} with $\tau_c^\vee(\mathcal{H}) = \|\mathcal{H}\|/3$) is at most three. It would be interesting to characterize all extremal hypergraphs for Theorem 2.1.

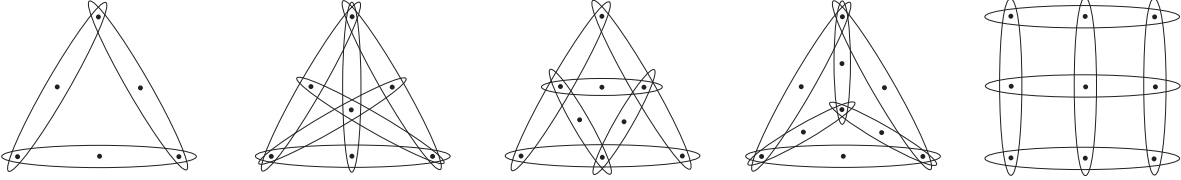


Figure 1: Extremal linear 3-uniform hypergraphs \mathcal{H} with $\tau_c^\vee(\mathcal{H}) = \|\mathcal{H}\|/3$.

The proof of Theorem 2.1 actually gives a recursive combinatorial algorithm for finding in polynomial time a FVS of size at most $\|\mathcal{H}\|/3$ on a linear 3-uniform hypergraph \mathcal{H} .

ALGORITHM 1: Feedback Vertex Sets of Linear 3-Uniform Hypergraphs

Input: Linear 3-uniform hypergraph $\mathcal{H} = (\mathcal{V}, \mathcal{E})$.

Output: $\text{ALG1}(\mathcal{H})$, which is a FVS of \mathcal{H} with cardinality at most $\|\mathcal{H}\|/3$.

1. **If** $|\mathcal{E}| \leq 2$ **Then** $\text{ALG1}(\mathcal{H}) \leftarrow \emptyset$
 2. **Else If** $\exists s \in \mathcal{V} \cup \mathcal{E}$ such that s is not contained in any cycle of \mathcal{H}
 3. **Then** $\text{ALG1}(\mathcal{H}) \leftarrow \text{ALG1}(\mathcal{H} \setminus s)$
 4. **If** $\exists s \in \mathcal{V}$ such that $d_{\mathcal{H}}(s) \geq 3$
 5. **Then** $\text{ALG1}(\mathcal{H}) \leftarrow \{s\} \cup \text{ALG1}(\mathcal{H} \setminus s)$
 6. **If** $\exists v \in \mathcal{V}$ such that $d_{\mathcal{H}}(v) = 1$
 7. **Then** Let $v_1 e_1 v_2 e_2 v_3 \dots e_k v_1$ be a cycle of \mathcal{H} such that $e_1 = \{v_1, v_2, v\}$
 8. $\text{ALG1}(\mathcal{H}) \leftarrow \{v_3\} \cup \text{ALG1}(\mathcal{H} \setminus \{e_1, e_2, e_3\})$
 9. Let $(\mathcal{V}_c, \mathcal{E}_c) = v_1 e_1 v_2 e_2 \dots v_k e_k v_1$ be a shortest cycle in \mathcal{H}
 10. For each $i \in [k]$, let $u_i \in \mathcal{V}_c$, $f_i \in \mathcal{E} \setminus \mathcal{E}_c$ be such that $\{u_i, v_i, v_{i+1}\} = e_i$, $u_i \in f_i$
 11. **If** $k \equiv 0 \pmod{3}$ **Then** $\text{ALG1}(\mathcal{H}) \leftarrow \{v_i : i \equiv 0 \pmod{3}, i \in [k]\} \cup \text{ALG1}(\mathcal{H} \setminus \mathcal{E}_c)$
 12. **If** $k \equiv 1 \pmod{3}$
 13. **Then If** $f_1 \neq f_3$ or $f_2 \neq f_4$
 14. **Then** Relabel vertices and edges if necessary to make $f_1 \neq f_3$
 15. $\mathcal{V}_* \leftarrow \{v_i : i \equiv 0 \pmod{3}, i \in [k] - [3]\}$
 16. $\text{ALG1}(\mathcal{H}) \leftarrow \{u_1, u_3\} \cup \mathcal{V}_* \cup \text{ALG1}(\mathcal{H} \setminus (\mathcal{E}_c \cup \{f_1, f_3\}))$
 17. **Else** $\text{ALG1}(\mathcal{H}) \leftarrow \{u_2, u_4\} \cup \text{ALG1}(\mathcal{H} \setminus (\mathcal{E}_c \cup \{f_1, f_2\}))$
 18. **If** $k \equiv 2 \pmod{3}$
 19. **Then** $\text{ALG1}(\mathcal{H}) \leftarrow \{u_1\} \cup \{v_i : i \equiv 1 \pmod{3}, i \in [k] - [3]\} \cup \text{ALG1}(\mathcal{H} \setminus (\mathcal{E}_c \cup \{f_1\}))$
 20. **Output** $\text{ALG1}(\mathcal{H})$
-

Note that Algorithm 1 never visits isolated vertices (it only scans along the edges of the current hypergraph). The number of iterations performed by the algorithm is upper bounded by $|\mathcal{E}|$. Since \mathcal{H} is 3-uniform,

the condition in any step is checkable in $O(|\mathcal{E}|^2)$ time. Any cycle in Step 7 or Step 9 can be found in $O(|\mathcal{E}|^2)$ time.¹ Thus Algorithm 1 runs in $O(|\mathcal{E}|^3)$ time.

Corollary 2.2. *Given any linear 3-uniform hypergraph \mathcal{H} , Algorithm 1 finds in $O(\|\mathcal{H}\|^3)$ time a FVS of \mathcal{H} with size at most $\|\mathcal{H}\|/3$.* \square

Lemma 2.3. *If $\mathcal{H} = (\mathcal{V}, \mathcal{E})$ is a connected linear 3-uniform hypergraph without cycles, then $|\mathcal{V}| = 2|\mathcal{E}| + 1$.*

Proof. We prove by induction on $|\mathcal{E}|$. The base case where $|\mathcal{E}| = 0$ is trivial. Inductively, we assume that $|\mathcal{E}| \geq 1$ and the lemma holds for all connected acyclic linear 3-uniform hypergraph of edges fewer than \mathcal{H} . Take arbitrary $e \in \mathcal{E}$. Since \mathcal{H} is connected, acyclic and 3-uniform, $\mathcal{H} \setminus e$ contains exactly three components $\mathcal{H}_i = (\mathcal{V}_i, \mathcal{E}_i)$, $i = 1, 2, 3$. Note that for each $i \in [3]$, hypergraph \mathcal{H}_i with $|\mathcal{E}_i| < |\mathcal{E}|$ is connected, linear, 3-uniform and acyclic. By the induction hypothesis, we have $|\mathcal{V}_i| = 2|\mathcal{E}_i| + 1$ for $i = 1, 2, 3$. It follows that $|\mathcal{V}| = \sum_{i=1}^3 |\mathcal{V}_i| = 2 \sum_{i=1}^3 |\mathcal{E}_i| + 3 = 2|\mathcal{E}| + 1$. \square

Given any hypergraph $\mathcal{H} = (\mathcal{V}, \mathcal{E})$, we can easily find a minimal (not necessarily minimum) FES in $O(|\mathcal{E}|^2)$ time: Go through the edges of the trivial FES \mathcal{E} in any order, and remove the edge from the FES immediately if the edge is redundant. The redundancy test can be implemented using Depth First Search.

Lemma 2.4. *Let $\mathcal{H} = (\mathcal{V}, \mathcal{E})$ be a linear 3-uniform hypergraph with p components. If \mathcal{F} is a minimal FES of \mathcal{H} , then $|\mathcal{F}| \leq 2|\mathcal{E}| - |\mathcal{V}| + p$. In particular, $\tau_c^\mathcal{E}(\mathcal{H}) \leq 2|\mathcal{E}| - |\mathcal{V}| + p$.*

Proof. Suppose that $\mathcal{H} \setminus \mathcal{F}$ contains exactly k components $\mathcal{H}_i = (\mathcal{V}_i, \mathcal{E}_i)$, $i = 1, \dots, k$. It follows from Lemma 2.3 that $|\mathcal{V}_i| = 2|\mathcal{E}_i| + 1$ for each $i \in [k]$. Thus $|\mathcal{V}| = \sum_{i \in [k]} |\mathcal{V}_i| = 2 \sum_{i \in [k]} |\mathcal{E}_i| + k = 2(|\mathcal{E}| - |\mathcal{F}|) + k$, which means $2|\mathcal{F}| = 2|\mathcal{E}| - |\mathcal{V}| + k$. To establish the lemma, it suffices to prove $k \leq |\mathcal{F}| + p$.

In case of $|\mathcal{F}| = 0$, we have $\mathcal{F} = \emptyset$ and $k = p = |\mathcal{F}| + p$. In case of $|\mathcal{F}| \geq 1$, suppose that $\mathcal{F} = \{e_1, \dots, e_{|\mathcal{F}|}\}$. Because \mathcal{F} is a minimal FES of \mathcal{H} , for each $i \in [|\mathcal{F}|]$, there is a cycle \mathcal{C}_i in $\mathcal{H} \setminus (\mathcal{F} \setminus \{e_i\})$ such that $e_i \in \mathcal{C}_i$, and $\mathcal{C}_i \setminus e_i$ is a path in $\mathcal{H} \setminus \mathcal{F}$ connecting two of the three vertices in e_i . Considering $\mathcal{H} \setminus \mathcal{F}$ being obtained from \mathcal{H} by removing $e_1, e_2, \dots, e_{|\mathcal{F}|}$ sequentially, for $i = 1, \dots, |\mathcal{F}|$, since $|e_i| = 3$, the presence of path $\mathcal{C}_i \setminus e_i$ implies that the removal of e_i can create at most one more component. Therefore we have $k \leq p + |\mathcal{F}|$ as desired. \square

Given a hypergraph $\mathcal{H} = (\mathcal{V}, \mathcal{E})$ with n vertices and m edges, let $M_{\mathcal{H}}$ be the $\mathcal{V} \times \mathcal{E}$ incidence matrix. From $M_{\mathcal{H}}$, we may construct a bipartite graph $G_{\mathcal{H}}$ with bipartition \mathcal{V}, \mathcal{E} such that there is an edge of $G_{\mathcal{H}}$ between $v \in \mathcal{V}$ and $e \in \mathcal{E}$ if and only if $v \in e$ in \mathcal{H} .

Suppose that \mathcal{H} is acyclic. It is easy to see that $G_{\mathcal{H}}$ is acyclic. Thus $M = M_{\mathcal{H}}$ falls within the class of *restricted totally unimodular* (RTUM) matrices defined by Yannakakis [10]. As the name indicates, RTUM matrices are all totally unimodular. Hence the total unimodularity and LP duality give the well-known result [11] that $\tau(\mathcal{H}) = \min\{\mathbf{1}^T \mathbf{x} : M^T \mathbf{x} \geq \mathbf{1}, x \geq 0\} = \max\{\mathbf{1}^T \mathbf{y} : M \mathbf{y} \leq \mathbf{1}, y \geq 0\} = \nu(\mathcal{H})$. Moreover, since M is RTUM, both a minimum transversal and a maximum matching of \mathcal{H} can be found in $O(n(m + n \log n) \log n)$ time using Yanakakis's combinatorial algorithm [10] based on the current best combinatorial algorithms for the b -matching problem and the maximum weighted independent set problem on a bipartite multigraph with n vertices and m edges, where the bipartite b -matching problem can be solved with the minimum-cost flow algorithm in $O(n \log n(m + n \log n))$ time (see Section 21.5 and Page 356 of [12]) and the maximum weighted independent set problem can be solved with maximum flow algorithm in $O(nm \log n)$ time (See Pages 300-301 of [10]).

Theorem 2.5 ([11, 10]). *Let \mathcal{H} be a hypergraph with n non-isolated vertices and m edges. If \mathcal{H} has no cycle, then $\tau(\mathcal{H}) = \nu(\mathcal{H})$, and a minimum transversal and a maximum matching of \mathcal{H} can be found in $O(n(m + n \log n) \log n)$ time.* \square

¹The shortest path between any pair of vertices can be find in $O(|\mathcal{E}|)$ time using breadth first search. A shortest cycle can be find by checking all $O(|\mathcal{E}|)$ possibilities.

3 Triangle packing and covering

This section establish several new sufficient conditions for Conjecture 1.1 as well as their algorithmic implications on finding minimum triangle covers. Section 3.1 deals with graphs of high triangle packing numbers. Section 3.2 investigates irreducible graphs with many edges. Section 3.3 discusses Erdős-Rényi graphs with high densities.

To each graph $G = (V, E)$, we associate a hypergraph $\mathcal{H}_G = (E, \mathcal{T}_G)$, referred to as *triangle hypergraph* of G , such that the vertices and edges of \mathcal{H}_G are the edges and triangles of G , respectively. Since G is simple, it is easy to see that \mathcal{H}_G is 3-uniform and linear, $\nu(\mathcal{H}_G) = \nu_t(G)$ and $\tau(\mathcal{H}_G) = \tau_t(G)$. Note that $\|\mathcal{H}_G\| = |\mathcal{T}_G| < \min\{|V|^3, |E|^3\}$, and $|E| \leq 3|\mathcal{T}_G|$ if G is irreducible, i.e., $\cup_{T \in \mathcal{T}_G} E(T) = E$. Note that the number of non-isolated vertices of \mathcal{H}_G is upper bounded by $3\|\mathcal{H}_G\| = 3|\mathcal{T}_G|$.

3.1 Graphs with many edge-disjoint triangles

We investigate Tuza's conjecture for graphs with large packing numbers, which are firstly compared with the number of triangles, and then with the number of edges.

Theorem 3.1. *If graph G and real number $c \in (0, 1]$ satisfy $\nu_t(G)/|\mathcal{T}_G| \geq c$, then a triangle cover of G with size at most $\frac{3c+1}{3c}\nu_t(G)$ can be found in $O(|\mathcal{T}_G|^3)$ time, which implies $\tau_t(G)/\nu_t(G) \leq \frac{3c+1}{3c}$.*

Proof. We consider the triangle hypergraph $\mathcal{H}_G = (E, \mathcal{T}_G)$ of G which is 3-uniform and linear. By Corollary 2.2, we can find in $O(|\mathcal{T}_G|^3)$ time a FVS \mathcal{S} of \mathcal{H}_G with $|\mathcal{S}| \leq |\mathcal{T}_G|/3$. Since $\nu(\mathcal{H}_G) = \nu_t(G) \geq c|\mathcal{T}_G|$, it follows that $|\mathcal{S}| \leq \nu(\mathcal{H}_G)/(3c)$. As $\mathcal{H}_G \setminus \mathcal{S}$ is acyclic, Theorem 2.5 enables us to find in $O(|\mathcal{T}_G|^2 \log^2 |\mathcal{T}_G|)$ time a minimum transversal \mathcal{R} of $\mathcal{H}_G \setminus \mathcal{S}$ such that $|\mathcal{R}| = \tau(\mathcal{H}_G \setminus \mathcal{S}) = \nu(\mathcal{H}_G \setminus \mathcal{S})$. We observe that $\mathcal{S} \cup \mathcal{R} \subseteq E$ and $G \setminus (\mathcal{S} \cup \mathcal{R})$ is triangle-free. Hence $\mathcal{S} \cup \mathcal{R}$ is a triangle cover of G with size

$$|\mathcal{S} \cup \mathcal{R}| \leq \frac{\nu(\mathcal{H}_G)}{3c} + \nu(\mathcal{H}_G \setminus \mathcal{S}) \leq \frac{3c+1}{3c}\nu(\mathcal{H}_G) = \frac{3c+1}{3c}\nu_t(G),$$

which proves the theorem. \square

The special case of $c = 1/3$ in the above theorem gives the following result providing a new sufficient condition for Tuza's conjecture.

Corollary 3.2. *If graph G satisfies $\nu_t(G)/|\mathcal{T}_G| \geq 1/3$, then $\tau_t(G)/\nu_t(G) \leq 2$.* \square

The condition $\nu_t(G) \geq |\mathcal{T}_G|/3$ in Corollary 3.2 applies, in some sense, only to the class of large scale sparse graphs (which, e.g., does not include complete graphs on four or more vertices). The mapping from the real number c in the condition $\nu_t(G) \geq c|\mathcal{T}_G|$ to the coefficient $\frac{3c+1}{3c}$ in the conclusion $\tau_t(G) \leq \frac{3c+1}{3c}\nu_t(G)$ of Theorem 3.1 shows the trade-off between conditions and conclusions. As in Corollary 3.2, $c = \frac{1}{3}$ maps to $\frac{3c+1}{3c} = 2$ hitting the boundary of Tuza's conjecture. It remains to study graphs G with $\nu_t(G)/|\mathcal{T}_G| < \frac{1}{3}$. The next theorem (Theorem 3.3) tells us that actually we only need to take care of graphs G with $\nu_t(G)/|\mathcal{T}_G| \in (\frac{1}{4} - \epsilon, \frac{1}{3})$, where ϵ can be any arbitrarily small positive number. So, in some sense, to solve Tuza's conjecture, we only have a gap of $\frac{1}{3} - \frac{1}{4} = \frac{1}{12}$ to be bridged. Interestingly, for $c = \frac{1}{4}$, we have $\frac{3c+1}{3c} = \frac{7}{3} = 2.333...$, which is much better than the best known general bound 2.87 due to Haxell [2]. Only when $c \leq \frac{1}{6}$ does $\frac{3c+1}{3c}$ state a trivial bound equal to or greater than 3.

Theorem 3.3. *If there exists some real $\delta > 0$ such that Conjecture 1.1 holds for every graph G with $\nu_t(G)/|\mathcal{T}_G| \geq 1/4 - \delta$, then Conjecture 1.1 holds for every graph.*

Proof. If $\delta \geq \frac{1}{4}$, the theorem is trivial. We consider $0 < \delta < \frac{1}{4}$. As the set of rational numbers is dense, we may assume $\delta \in \mathbb{Q}$ and $1/4 - \delta = i/j$ for some $i, j \in \mathbb{N}$. Therefore $i/j < 1/4$ gives $4i + 1 \leq j$, i.e., $4 + 1/i \leq j/i$. It remains to prove that for any graph G with $\nu_t(G) < (i/j)|\mathcal{T}_G|$ there holds $\tau_t(G) \leq 2\nu_t(G)$.

Write k for the positive integer $i|\mathcal{T}_G| - j \cdot \nu_t(G)$. Let G' be the disjoint union of G and k copies of K_4 . Clearly, $|\mathcal{T}_{G'}| = |\mathcal{T}_G| + k|\mathcal{T}_{K_4}| = |\mathcal{T}_G| + 4k$, $\tau_t(G') = \tau_t(G) + k \cdot \tau_t(K_4) = \tau_t(G) + 2k$ and $\nu_t(G') = \nu_t(G) + k \cdot \nu_t(K_4) = \nu_t(G) + k$. It follows that

$$\begin{aligned} (i/j)|\mathcal{T}_{G'}| &= (i/j)(|\mathcal{T}_G| + 4k) \\ &= (i/j)((k + j \cdot \nu_t(G))/i + 4k) \\ &= (i/j)(j \cdot \nu_t(G)/i + (4 + 1/i)c) \\ &\leq \nu_t(G) + k \\ &= \nu_t(G') \end{aligned}$$

where the inequality is guaranteed by $4 + 1/i \leq j/i$. So $\nu_t(G') \geq (1/4 - \delta)|\mathcal{T}_{G'}|$ together with the hypothesis of the theorem implies $\tau_t(G') \leq 2\nu_t(G')$, i.e., $\tau_t(G) + 2k \leq 2(\nu_t(G) + k)$, giving $\tau_t(G) \leq 2\nu_t(G)$ as desired. \square

In the proof of the above theorem, the property of K_4 that $\nu_t(K_4)/|\mathcal{T}_{K_4}| = 1/4$ and $\tau_t(K_4)/\nu_t(K_4) = 2$ plays an important role. It helps to reduce the general Tuza's conjecture to the special case where $\nu_t(G) \geq (1/4 - \delta)|\mathcal{T}_G|$.

The sufficient condition that compares the triangle packing number with the number of edges is based on the fact that every simple graph $G = (V, E)$ has a bipartite subgraph of at least $|E|/2$ edges, which can be found in polynomial time. Since this subgraph does not contain any triangle, we deduce that $\tau_t(G) \leq |E|/2$, which implies the following result.

Corollary 3.4. *If $G = (V, E)$ is a graph such that $\nu_t(G)/|E| \geq c$ for some $c > 0$, then $\tau_t(G)/\nu_t(G) \leq 1/(2c)$. In particular, if $\nu_t(G)/|E| \geq 1/4$, then $\tau_t(G)/\nu_t(G) \leq 2$. \square*

Thus if $\nu_t(G)/|E| \geq c$ for some $c > 0$, then a triangle cover of G with size at most $\nu_t(G)/(2c)$ can be found in polynomial time. Complementary to Corollary 3.2 whose condition mainly takes care of sparse graphs, the second statement of Corollary 3.4 applies to many dense graphs, including complete graphs on 25 or more vertices.

Similar to Corollary 3.2 and Theorem 3.3, by which our future investigation space on Tuza's conjecture shrinks to interval $(\frac{1}{4} - \epsilon, \frac{1}{3})$ w.r.t. $\nu_t(G)/|\mathcal{T}_G|$, Corollary 3.4 and the following Theorem 3.5 narrow the interval w.r.t. $\nu_t(G)/|E|$ to $(\frac{1}{5} - \epsilon, \frac{1}{4})$. Moreover, when taking $c = \frac{1}{5}$ in Corollary 3.4, we obtain $\frac{1}{2c} = 2.5$, still better than Haxell's general bound 2.87 [2].

Theorem 3.5. *If there exists some real $\delta > 0$ such that Conjecture 1.1 holds for every graph G with $\nu_t(G)/|E| \geq 1/5 - \delta$, then Conjecture 1.1 holds for every graph.*

Proof. We use the similar trick to that in proving Theorem 3.3; we add a number of complete graphs on five (instead of four) vertices. We may assume $\delta \in (0, \frac{1}{5}) \cap \mathbb{Q}$ and $1/5 - \delta = i/j$ for some $i, j \in \mathbb{N}$. Therefore $i/j < 1/5$ and the integrality of i, j imply $5 + 1/i \leq j/i$. To prove Tuza's conjecture for each graph G with $\nu_t(G) < (i/j)|E|$, we write $k = i|E| - j \cdot \nu_t(G) \in \mathbb{N}$. Let $G' = (V', E')$ be the disjoint union of G and k copies of K_5 's. Then $|E'| = |E| + 10k$, $\tau_t(G') = \tau_t(G) + k \cdot \tau_t(K_5) = \tau_t(G) + 4k$, $\nu_t(G') = \nu_t(G) + k \cdot \nu_t(K_5) = \nu_t(G) + 2k$, and

$$(i/j)|E'| = (i/j)(|E| + 10k) = (i/j)(j \cdot \nu_t(G)/i + (10 + 1/i)k) \leq \nu_t(G) + 2k = \nu_t(G')$$

where the inequality is guaranteed by $10 + 1/i \leq 2j/i$. So $\nu_t(G') \geq (1/5 - \delta)|E'|$ together with the hypothesis of the theorem implies $\tau_t(G') \leq 2\nu_t(G')$, i.e., $\tau_t(G) + 4k \leq 2(\nu_t(G) + 2k)$, giving $\tau_t(G) \leq 2\nu_t(G)$ as desired. \square

3.2 Graphs with many edges on triangles

Each graph has a unique maximum irreducible subgraph. Tuza's conjecture is valid for a graph if and only if the conjecture is valid for its maximum irreducible subgraph. In this section, we study sufficient conditions for Tuza's conjecture on irreducible graphs that bound the number of edges below in terms of the number of triangles.

Theorem 3.6. *If $G = (V, E)$ is an irreducible graph such that $|E|/|\mathcal{T}_G| \geq 2$, then a triangle cover of G with cardinality at most $2\nu_t(G)$ can be found in $O(|\mathcal{T}_G|^2 \log^2 |\mathcal{T}_G|)$ time, which implies $\tau_t(G)/\nu_t(G) \leq 2$.*

Proof. Suppose that the linear 3-uniform hypergraph $\mathcal{H} = (E, \mathcal{T}_G)$ associated to G has exactly p components. By Lemma 2.4, we can find in $O(|\mathcal{T}_G|^2)$ time a minimal FES \mathcal{F} of \mathcal{H} such that $|\mathcal{F}| \leq 2|\mathcal{T}_G| - |E| + p \leq p$. Since G is irreducible, we see that \mathcal{H} has no isolated vertices, i.e., every component of \mathcal{H} has at least one edge. Thus $\nu(\mathcal{H}) \geq p \geq |\mathcal{F}|$. For the acyclic hypergraph $\mathcal{H} \setminus \mathcal{F}$, By Lemma 2.5 we may find in $O(|\mathcal{T}_G|^2 \log^2 |\mathcal{T}_G|)$ time a minimum transversal \mathcal{R} of $\mathcal{H} \setminus \mathcal{F}$ such that

$$|\mathcal{R}| = \tau(\mathcal{H} \setminus \mathcal{F}) = \nu(\mathcal{H} \setminus \mathcal{F}).$$

Observe that $\mathcal{R} \subseteq E$ and $\mathcal{F} \subseteq \mathcal{T}_G$. If $\mathcal{F} = \emptyset$, set $\mathcal{S} = \emptyset$, else for each $F \in \mathcal{F}$, take $e_F \in E$ with $e_F \in F$, and set $\mathcal{S} = \{e_F : F \in \mathcal{F}\}$. It is clear that $\mathcal{R} \cup \mathcal{S}$ is a transversal of \mathcal{H} (i.e., a triangle cover of G) with cardinality $|\mathcal{R} \cup \mathcal{S}| \leq \nu(\mathcal{H} \setminus \mathcal{F}) + |\mathcal{F}| \leq 2\nu(\mathcal{H}) = 2\nu_t(G)$, establishing the theorem. \square

We observe that the graphs G which consist of a number of triangles sharing a common edge satisfy $|E(G)| \geq 2|\mathcal{T}_G|$, but $\nu_t(G) < |\mathcal{T}_G|/3$ when $|\mathcal{T}_G| \geq 4$. So in some sense, Theorem 3.6 works a supplement of Corollary 3.2 for sparse graphs.

A multigraph is *series-parallel* if and only if it can be constructed from a single edge by iteratively performing the *D-Operation* of doubling an edge and/or the *S-Operation* of subdividing an edge. A graph is a *2-tree* if and only if it can be constructed from a single edge by iteratively performing the *DS-Operation* of doubling an edge and subdivide the new edge with a new vertex. A subgraph of a 2-tree is called a *partial 2-tree*. It is well-known that a (simple) graph is a partial 2-tree if and only if all of its maximal 2-connected subgraphs are series-parallel [13]. Thus, a series-parallel (simple) graph is a partial 2-tree. In the following we show that every partial 2-tree G satisfies $|E(G)| \geq 2|\mathcal{T}_G|$.

Corollary 3.7. *If $G = (V, E)$ is a partial 2-tree, then a triangle cover of G with cardinality at most $2\nu_t(G)$ can be found in $O(|E|^2 \log^2 |E|)$ time.*

Proof. In $O(|E|^2)$ time, we may remove from G all edges that are not contained in any triangles. The resulting graph is still a partial 2-tree. So we may assume without loss of generality that G is irreducible. Since each triangle of G is contained a unique maximal 2-connected subgraph of G , we may further assume that G is 2-connected. It follows that G is series-parallel. Since G is simple, it can be constructed from a single edge by iteratively performing the S-Operation and/or the DS-Operation. The S-Operation increases the number of edges and does not change the number of triangles, while the DS-Operation increases the number of edges by 2 and the number of triangles by 1. Therefore, we have $|E| \geq 2|\mathcal{T}_G|$. The conclusion follows from Theorem 3.6. \square

Note that partial 2-trees are K_4 -free planar graphs. The validity of Tuza's conjecture on partial 2-trees has been verified in [3, 6]. The 2-approximation algorithm for finding a minimum triangle cover in planar graphs implied by Tuza's proof [3] runs in $O(|E|)$ time.

Along the same line as in the previous subsection, regarding Tuza's conjecture on graph G , Theorem 3.6 and the following Theorem 3.8 jointly narrow the interval w.r.t. $|E(G)|/|\mathcal{T}_G|$ to $(1.5 - \epsilon, 2)$ for future study.

Theorem 3.8. *If there exists some real $\delta > 0$ such that Conjecture 1.1 holds for every irreducible graph $G = (V, E)$ with $|E|/|\mathcal{T}_G| \geq 3/2 - \delta$, then Conjecture 1.1 holds for every irreducible graph (and therefore every graph).*

Proof. Again we apply the trick of adding copies of K_4 . We may assume $\delta \in (0, 3/2) \cap \mathbb{Q}$ and $3/2 - \delta = i/j$ for some $i, j \in \mathbb{N}$. Therefore $2i + 1 \leq 3j$, implying $(i/j)(4 + 1/i) \leq 6$.

For any irreducible graph G with $|E| < (i/j)|\mathcal{T}_G|$, we write $k = i|\mathcal{T}_G| - j|E| \in \mathbb{N}$. Let G' be the disjoint union of G and k copies of K_4 . Then G' is irreducible, and

$$(i/j)|\mathcal{T}_{G'}| = (i/j)(|\mathcal{T}_G| + 4k) = (i/j)(j|E|/i + (4 + 1/i)k) \leq |E| + 6k = |E'|.$$

It follows from the hypothesis of the theorem that $\tau_t(G') \leq 2\nu_t(G')$, i.e., $\tau_t(G) + 2k \leq 2(\nu_t(G) + k)$, giving $\tau_t(G) \leq 2\nu_t(G)$ as desired. \square

3.3 Erdős-Rényi graphs with high densities

Let n be a positive integer, and let $p \in [0, 1]$. The Erdős-Rényi random graph model [14] is a probability space over the set $\mathcal{G}(n, p)$ of graphs $G = (V, E)$ on the vertex set $V = \{1, \dots, n\}$, where an edge between vertices i and j is included in E with probability p independent from every other possible edge, i.e.,

$$\Pr[ij \in E] = p \text{ for each pair of distinct } i, j \in V.$$

The $\mathcal{G}(n, p)$ model is often used in the probabilistic method for tackling problems in various areas such as graph theory and combinatorial optimization.

The following result on the triangle packing numbers of complete graphs [15] is useful in deriving a good estimation for the triangle packing numbers of graphs in $\mathcal{G}(n, p)$.

Theorem 3.9 ([15]). $\nu_t(K_n) = |E(K_n)|/3$ if and only if $n \equiv 1, 3 \pmod{6}$. □

Theorem 3.10. Suppose that $p > \sqrt{3}/2$ and $G = (V, E) \in \mathcal{G}(n, p)$. Then $\Pr[\nu_t(G) \geq |E|/4] = 1 - o(1)$ and $\Pr[\tau_t(G) \leq 2\nu_t(G)] = 1 - o(1)$.

Proof. Let K_n denote the complete graph on V . For each edge $e \in K_n$, let X_e be the indicator variable satisfying: $X_e = 1$ if $e \in E$ and $X_e = 0$ otherwise. Thus $\mathbf{E}[X_e] = p$, $X = \sum_{e \in K_n} X_e = |E|$, $\mathbf{E}[X] = n(n-1)p/2$. Since $X_e, e \in K_n$, are independent 0-1 variables, by Chernoff Bounds, for each $\epsilon \in (0, 1]$, $\Pr[X > (1 + \epsilon)\mathbf{E}[X]] \leq \exp(-\epsilon^2\mathbf{E}[X]/3) = \exp(-\epsilon^2n(n-1)p/6) = o(1)$. So

$$\Pr[X \leq (1 + \epsilon)\mathbf{E}[X]] = \Pr[X \leq (1 + \epsilon)n(n-1)p/2] = 1 - o(1).$$

On the other hand, by Theorem 3.9, we can make K_n have an edge-disjoint triangle decomposition by deleting at most three vertices, which implies that $\nu_t(K_n)$ is lower bounded by $k = \lceil (n-3)(n-4)/6 \rceil$. Thus we can take k edge-disjoint triangles T_1, \dots, T_k from K_n . For each $i \in [k]$, let Y_i be the indicator variable satisfying: $Y_i = 1$ if $T_i \subseteq G$ and $Y_i = 0$ otherwise. Note that $\mathbf{E}[Y_i] = p^3$ for each $i \in [k]$, $\nu_t(G) \geq Y = \sum_{i=1}^k Y_i$ and $\mathbf{E}[Y] = kp^3$. Because T_1, \dots, T_k are edge-disjoint, Y_1, \dots, Y_k are independent 0-1 variables. By Chernoff Bounds, for each $\epsilon \in (0, 1)$, $\Pr[Y < (1 - \epsilon)\mathbf{E}[Y]] \leq \exp(-\epsilon^2\mathbf{E}[Y]/2) \leq \exp(-\epsilon^2(n-3)(n-4)p^3/12) = o(1)$. Thus

$$\Pr[\nu_t(G) \geq (1 - \epsilon)(n-3)(n-4)p^3/6] \geq \Pr[\nu_t(G) \geq (1 - \epsilon)kp^3] \geq \Pr[Y \geq (1 - \epsilon)\mathbf{E}[Y]] = 1 - o(1).$$

Recall that $p > \sqrt{3}/2$. We can take $\epsilon \in (0, 1)$ such that $\lim_{n \rightarrow \infty} \frac{(1-\epsilon)(n-3)(n-4)p^3/6}{(1+\epsilon)n(n-1)p/8} = \frac{4p^2(1-\epsilon)}{3(1+\epsilon)} > 1$. So for sufficient large n , we always have $(1 - \epsilon)(n-3)(n-4)p^3/6 > (1 + \epsilon)n(n-1)p/8$. Since we have $\nu_t(G) \geq (1 - \epsilon)(n-3)(n-4)p^3/6$ with probability $1 - o(1)$ and have $|E| = X \leq (1 + \epsilon)n(n-1)p/2$ with probability $1 - o(1)$, we obtain $\nu_t(G) \geq |E|/4$ with probability $1 - o(1)$. It follows from Corollary 3.4 that $\Pr[\tau_t(G) \leq 2\nu_t(G)] = 1 - o(1)$. □

4 Conclusion

Using tools from hypergraphs, we design polynomial-time algorithms for finding a small triangle covers in graphs, which particularly imply several sufficient conditions for Tuza's conjecture (Conjecture 1.1).

Triangle packing and covering. In this paper, we have established new sufficient conditions $\nu_t(G)/|\mathcal{T}_G| \geq 1/3$ and $|E|/|\mathcal{T}_G| \geq 2$ for Tuza's conjecture on packing and covering triangles in graphs G . We prove the sufficiency by designing polynomial-time combinatorial algorithms for finding a triangle cover of G whose cardinality is upper bounded by $2\nu_t(G)$. The high level idea of these algorithms is to remove *some edges* from G so that the triangle hypergraph of the remaining graph is *acyclic* (see the proofs of Theorems 2.1 and 3.6), which guarantees that the remaining graph has equal triangle covering number and triangle packing number, and a minimum triangle cover of the remaining graph is computable in polynomial time (see Theorem 2.5).

It is well-known that the acyclic condition in Theorem 2.5 could be weakened to odd-cycle-freeness [10]. So the lower bound $1/3$ and 2 in the sufficient conditions could be (significantly) improved if we can remove (much) *fewer edges* from G such that the triangle hypergraph of the remaining graph is *odd-cycle free*.

In view of Theorems 3.3, 3.5 and 3.8, the study on the graphs G satisfying $\nu_t(G)/|\mathcal{T}_G| \geq 1/4$ or $\nu_t(G)/|E| \geq 1/5$ or $|E|/|\mathcal{T}_G| \geq 3/2$ might suggest more insight and foresight for resolving Tuza's conjecture. These graphs are critical in the sense that they are standing on the border of the resolution.

Let us pay more attention to *extremal graphs* G which satisfy Tuza's conjecture with tight ratio $\tau_t(G)/\nu_t(G) = 2$. Actually, from Theorem 3.1, Corollary 3.4 and Theorem 3.6, we can get a nice observation: for every irreducible extremal graph $G = (V, E)$, the following three inequalities hold on: $\nu_t(G)/|\mathcal{T}_G| \leq 1/3$, $\nu_t(G)/|E| \leq 1/4$, and $|E|/|\mathcal{T}_G| \leq 2$. Gregory J. Puleo first notices this observation.

Another intermediate step towards resolving Tuza's conjecture is investigating its validity for the classical Erdős-Rényi random graph model $\mathcal{G}(n, p)$. In this paper, we have shown that Tuza's conjecture holds with high probability for graphs in $\mathcal{G}(n, p)$ when $p > \sqrt{3}/2$. It would be nice to prove the same result for $p \in (0, \sqrt{3}/2]$.

The generalization to linear 3-uniform hypergraphs. Our work has shown very close relations between triangle packing and covering in graphs and edge (resp. cycle) packing and covering in linear 3-uniform hypergraphs. The theoretical and algorithmic results on linear 3-uniform hypergraphs (Corollary 2.2 and Lemma 2.4) are crucial for us to establish sufficient conditions for Tuza's conjecture, and to find in strongly polynomial time a "small" triangle cover under the conditions (see Corollary 3.2 and Theorem 3.6). Recall that, for any graph G , its triangle hypergraph \mathcal{H}_G is linear 3-uniform, and Tuza's conjecture is equivalent to $\tau(\mathcal{H}_G) \leq 2\nu(\mathcal{H}_G)$. As a natural generalization, one may ask: Does $\tau(\mathcal{H}) \leq 2\nu(\mathcal{H})$ hold for all linear 3-uniform hypergraphs \mathcal{H} ? It is easy to see that $\{\mathcal{H}_G : G \text{ is a graph}\}$ is *properly* contained in the set of linear 3-uniform hypergraphs. Unfortunately, Zbigniew Lonc pointed out there is a simple negative example: The Fano projective plane is an example of a linear 3-uniform hypergraph whose matching number is 1 and transversal number is 3 (See Figure 2). Last but not the least, the arguments in the paper have actually proved the following stronger result.

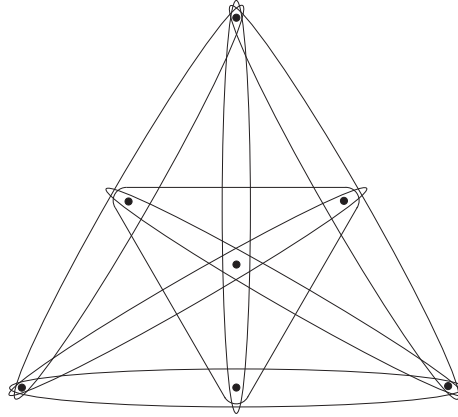


Figure 2: The Fano projective plane

Theorem 4.1. *Let $\mathcal{H} = (\mathcal{V}, \mathcal{E})$ be a linear 3-uniform hypergraph without isolated vertices. Then a transversal of \mathcal{H} with cardinality at most $2\nu(\mathcal{H})$ can be found in polynomial time, which implies $\tau(\mathcal{H}) \leq 2\nu(\mathcal{H})$, if one of the following conditions is satisfied: (i) $\nu(\mathcal{H})/|\mathcal{E}| \geq \frac{1}{3}$, (ii) $|\mathcal{V}|/|\mathcal{E}| \geq 2$. \square*

Comparing the above result on linear 3-uniform hypergraphs \mathcal{H} with its counterpart on graphs presented in Theorem 1.2, one might notice that the condition on the lower bound of $\nu(\mathcal{H})/|\mathcal{V}|$ is missing. This reason is that we do not have a nontrivial constant upper bound on $\tau(\mathcal{H})/|\mathcal{V}|$.

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